

# An integrable hierarchy, parametric solution and traveling wave solution

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## Abstract

This paper gives an integrable hierarchy of nonlinear evolution equations. In this hierarchy there are the following representative equations:

$$\begin{aligned} u_t &= \partial_x^5 u^{-\frac{2}{3}}, \\ u_t &= \partial_x^5 \frac{(u^{-\frac{1}{3}})_{xx} - 2(u^{-\frac{1}{6}})_x^2}{u}, \\ u_{xxt} + 3u_{xx}u_x + u_{xxx}u &= 0. \end{aligned}$$

The first two are in the positive order hierarchy while the 3rd one is in the negative order hierarchy. The whole hierarchy is shown integrable through solving a key  $3 \times 3$  matrix equation. The  $3 \times 3$  Lax pairs and their adjoint representations are nonlinearized to be two Liouville-integrable canonical Hamiltonian systems. Based on the integrability of  $6N$ -dimensional systems we give the parametric solution of the positive hierarchy. In particular, we obtain the parametric solution of the equation  $u_t = \partial_x^5 u^{-\frac{2}{3}}$ . Moreover, we give the traveling wave solution (TWS) of the above three equations. The TWSs of the first two equations have singularity and look like cusp (cusp-like), but the TWS of the 3rd one is continuous. For the 5th-order equation, its parametric solution can not include its singular TWS. We also analyse the Gaussian initial solutions for the equations  $u_t = \partial_x^5 u^{-\frac{2}{3}}$ , and  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ . One is stable, the other not. Finally, we extend the equation  $u_t = \partial_x^5 u^{-\frac{2}{3}}$  to a large class of equations  $u_t = \partial_x^l u^{-m/n}$ ,  $l \geq 1$ ,  $n \neq 0$ ,  $m, n \in \mathbb{Z}$ , which still have the singular cusp-like traveling wave solutions.

**Keywords** Hamiltonian system, Matrix equation, Zero curvature representation, Parametric solution, Traveling wave solution.

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## 1 Introduction

The inverse scattering transformation (IST) method plays a very important role in the investigation of integrable nonlinear evolution equations (NLEEs) [9]. This method has been successfully applied to solve the integrable NLEEs in the form of soliton solutions. These NLEEs include the well-known KdV equation [15], which is related to a 2nd order operator (i.e. Hill operator) spectral problem [16, 18], the remarkable AKNS equations [1, 2], which is associated with the Zakharov-Shabat (ZS) spectral problem [23], and other higher dimensional integrable equations.

In the theory of integrable system, it is significant for us to search for as many new integrable evolution equations as possible. Kaup [12] studied the inverse scattering problem for cubic eigenvalue equations of the form  $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$ , and showed a 5th order partial differential equation (PDE)  $Q_t + Q_{xxxxx} + 30(Q_{xxx}Q + \frac{5}{2}Q_{xx}Q_x) + 180Q_xQ^2 = 0$  (called the KK equation) integrable. Afterwards, Kuper-schmidt [14] constructed a super-KdV equation and presented the integrability of the equation through giving bi-Hamiltonian property and Lax form. Very recently, Degasperis and Procesi [7] proposed a new integrable equation (called DP equation) with the soliton solution of peakon type. The DP equation is an extension of the Camassa-Holm (CH) equation [4], and is proven to be associated with a 3rd order spectral problem [6]:  $\psi_{xxx} = \psi_x - \lambda m\psi$  and to have some relationship to a canonical Hamiltonian system under a new nonlinear Poisson bracket (called Peakon Bracket) [10]. In [11], we extended the DP equation to an integrable hierarchy and deal with its parametric solution and peaked stationary solutions.

In Ref. [6], the authors studied the DP equation, of a similar form to the Camassa-Holm shallow water wave equation, and proved the exact integrability of this equation by constructing its Lax pair. The DP equation is related to a negative flow in the Kaup-Kupershmidt hierarchy via a reciprocal transformation. The infinite sequence of conserved quantities is derived together with a proposed bi-Hamiltonian structure. The equation admits exact solutions in the form of a superposition of multi-peakons, and is compared with the analogous results for Camassa-Holm peakons.

The present work is motivated on the basis of the eigenvalue problem  $\psi_x - \alpha^2\psi_{xxx} = \alpha^2\lambda m\psi$ , which was introduced by Degasperis, Holm and Hone [6]. Here, we consider the limit case of  $\alpha$  going to infinity. That is, we get the 3rd order spectral problem  $\psi_{xxx} = -\lambda m\psi$ . In this paper, starting from that problem, we give an integrable hierarchy, and through solving a key matrix equation we explicitly

provide the Lax pairs for the whole hierarchy. The following equations

$$u_t = \partial_x^5 u^{-\frac{2}{3}}, \quad (1)$$

$$u_t = \partial_x^5 \frac{(u^{-\frac{1}{3}})_{xx} - 2(u^{-\frac{1}{6}})_x^2}{u}; \quad (2)$$

$$u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0, \quad (3)$$

are three representatives in the hierarchy. The first equation is a reduction of some  $2+1$  dimensional equation [13]. The second one is new, and the third one is actually twice derivatives in  $x$  of the Riemann shock equation [6].

Konopelchenko and Dubrovsky [13] already pointed out equation (1) is integrable from the reduction view point [13], but did not discuss the spectral problem and the solution of the equation. Here we deal with its spectral problem and representation of solution from the constraint view point. We give the parametric solutions for the hierarchy, particularly for equation (1). Furthermore, we obtain the traveling wave solution (TWS) for equations (1), (2), and (3). The first two look like a class of cusp soliton solutions (called ‘cusp-like’, but not cusp soliton [22]). The TWSs of equations (1) and (2) have singularity, but the TWS of equation (3) is continuous. Additionally, for the 5th-order equation (1), its smooth parametric solution can not include its singular TWS. Equation (3) has the compacton-like and parabolic cylinder solutions. We also analyse the Gaussian initial solutions for equations  $u_t = \partial_x^5 u^{-\frac{2}{3}}$  and  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ . The former is stable, and the latter not (see Figures 7, 1 - 5). Finally, we extend the equation  $u_t = \partial_x^5 u^{-\frac{2}{3}}$  to a large class of equations  $u_t = \partial_x^l u^{-m/n}$ ,  $l \geq 1$ ,  $n \neq 0$ ,  $m, n \in \mathbb{Z}$ , which still have the singular cusp-like traveling wave solutions.

The whole paper is organized as follows. Next section is saying how to connect the above three equations to a spectral problem and how to cast them into a new hierarchy of NLEEs. In section 3, we construct the zero curvature representations for this new hierarchy through solving a key  $3 \times 3$  matrix equation. In particular, we obtain the Lax pair of equations (1), (2), (3), and therefore they are integrable. In section 4, we show that the 3rd order spectral problem related to the above three equations is nonlinearized as a completely integrable Hamiltonian system under some constraint in  $\mathbb{R}^{6N}$ . In section 5 we give the parametric solution for the positive order hierarchy of NLEEs. We particularly get the parametric solution of equation (1). Moreover, in section 6 we obtain the traveling wave solutions for equations (1), (2), and (3), and we also analyse the Gaussian initial solutions for the equations  $u_t = \partial_x^5 u^{-\frac{2}{3}}$  and  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ . We propose a family of new equations  $u_t = \partial_x^l u^{-m/n}$ ,  $l \geq 1$ ,  $n \neq 0$ ,  $m, n \in \mathbb{Z}$ , which still have the singular cusp-like traveling wave solutions. Finally, in section 7 we give some conclusions.

## 2 Spectral problems and a new hierarchy

Let us consider the following 3rd order spectral problem

$$\psi_{xxx} = -\lambda u \psi \quad (4)$$

and its adjoint problem

$$\psi_{xxx}^* = \lambda u \psi^*. \quad (5)$$

Take  $u \rightarrow u + \epsilon \delta u$ ,  $\lambda \rightarrow \lambda + \epsilon \delta \lambda$ , and  $\frac{\partial}{\partial \epsilon}|_{\epsilon=0}$ . Then, we have their functional gradient  $\frac{\delta \lambda}{\delta u}$  with respect to the potential  $u$

$$\frac{\delta \lambda}{\delta u} = \frac{\lambda \psi \psi^*}{E} \equiv \frac{\nabla \lambda}{E}, \quad (6)$$

where

$$\begin{aligned} \nabla \lambda &= \lambda \psi \psi^*, \\ E &= \int_{\Omega} u \psi \psi^* dx = \text{constant}, \end{aligned} \quad (7)$$

and  $\Omega = (-\infty, \infty)$  or  $\Omega = (0, T)$ . In this procedure, we need the boundary conditions of  $u$  decaying at infinities or of  $u$  being periodic with period  $T$ . Usually, we compute the functional gradient  $\frac{\delta \lambda}{\delta u}$  of the eigenvalue  $\lambda$  with respect to the potential  $u$  by using the method in Refs. [5, 21].

Through doing five times derivatives of Eq. (7), we find

$$\begin{aligned} (\nabla \lambda)_{xxxxx} &= -3\lambda^2(2u\partial + \partial u)(\psi\psi_x^* - \psi^*\psi_x), \\ (\psi\psi_x^* - \psi^*\psi_x)_{xxx} &= (u\partial + 2\partial u)\nabla \lambda, \end{aligned}$$

which directly lead to

$$K\nabla \lambda = \lambda^2 J \nabla \lambda, \quad (8)$$

where

$$K = \partial^5, \quad (9)$$

$$J = -3(2u\partial + \partial u)\partial^{-3}(u\partial + 2\partial u). \quad (10)$$

**Hint:** Here we do not care about the Hamiltonian properties of the operators  $K, J$ , but need

$$\begin{aligned} K^{-1} &= \partial^{-5}, \\ J^{-1} &= -\frac{1}{27}u^{-2/3}\partial^{-1}u^{-1/3}\partial^3u^{-1/3}\partial^{-1}u^{-2/3}. \end{aligned}$$

They yield

$$\mathcal{L} = J^{-1}K = -\frac{1}{27}u^{-2/3}\partial^{-1}u^{-1/3}\partial^3u^{-1/3}\partial^{-1}u^{-2/3}\partial^5, \quad (11)$$

$$\mathcal{L}^{-1} = K^{-1}J = -3\partial^{-5}(2u\partial + \partial u)\partial^{-3}(u\partial + 2\partial u). \quad (12)$$

By this pair of operators we define the hierarchy of nonlinear evolution equations associated with the spectral problems (4) and (5). Let  $G_0 \in \text{Ker } J = \{G \in C^\infty(\mathbb{R}) \mid JG = 0\}$  and  $G_{-1} \in \text{Ker } K = \{G \in C^\infty(\mathbb{R}) \mid KG = 0\}$ . We define the Lenard sequence

$$G_j = \begin{cases} \mathcal{L}^j \cdot G_0, & j \geq 0, j \in \mathbb{Z} \\ \mathcal{L}^{j+1} \cdot G_{-1}, & j < 0, j \in \mathbb{Z}. \end{cases} \quad (13)$$

where  $\mathcal{L} = J^{-1}K$  is called the recursion operator. Therefore we produce a new hierarchy of nonlinear evolution equations (NLEEs):

$$u_{t_k} = JG_k, \quad \forall k \in \mathbb{Z}. \quad (14)$$

Apparently, this hierarchy includes the positive order ( $k \geq 0$ ) and the negative order ( $k < 0$ ) cases. Let us now give several representative equations in the hierarchy (14).

- Choosing  $G_{-1} = \frac{1}{6} \in \text{Ker } K$  yields the first equation in the negative hierarchy:

$$u_t + vu_x + 3v_xu = 0, \quad u = v_{xx}. \quad (15)$$

This equation is actually:  $v_{xxt} + 3v_{xx}v_x + v_{xxx}v = 0$  which is equivalent to  $\partial^2(v_t + vv_x) = 0$ . Obviously,  $v = c_1x + c_0$  ( $c_1, c_0$  are two constants) is a special solution of this equation. In section 6, we will study its traveling wave solution.

The second equation in the negative hierarchy is:

$$\begin{aligned} u_{t_{-2}} + 3(u_xw + 3uw_x) &= 0, \\ w_{xxx} + \frac{9}{2}(2u_xv + 3uv_x) &= 0, \\ u &= 3(\sqrt{v_{xx}})_{xx}. \end{aligned}$$

- Choosing  $G_0 = u^{-\frac{2}{3}} \in \text{Ker } J$  leads to the second equation in the positive hierarchy:

$$u_t = \partial_x^5 u^{-\frac{2}{3}}. \quad (16)$$

Konopelchenko and Dubrovsky ever pointed out that this equation is integrable and is a reduction of some  $2 + 1$  dimensional equation [13]. But they

did not study the solution of this equation. In the following, we study the relation between this equation and finite-dimensional integrable system and will find that it has parametric solution as well as the traveling wave solution which looks like a cusp.

The third equation in the positive hierarchy is:

$$\begin{aligned} u_{t_2} &= -\frac{1}{27}(u^{-2/3}v)_{5x}, \\ v_x &= u^{-1/3}(u^{-1/3}w)_{3x}, \\ w_x &= u^{-2/3}(u^{-2/3}w)_{5x}. \end{aligned}$$

- Choosing another element  $G_0 = \frac{(u^{-\frac{1}{3}})_{xx} - 2(u^{-\frac{1}{6}})_x^2}{u} \in \text{Ker } J$  gives the following representative equation in the positive hierarchy:

$$u_t = \partial_x^5 \frac{(u^{-\frac{1}{3}})_{xx} - 2(u^{-\frac{1}{6}})_x^2}{u}. \quad (17)$$

This equation also has a singular cusp-like traveling wave solution.

Of course, we may produce further nonlinear equations by selecting other elements from the kernels of  $J, K$ . In the following, we will see that all equations in the hierarchy (14) are integrable. Particularly, **the above three equations (15), (16), (17) are integrable.**

### 3 Zero curvature representations

Letting  $\psi = \psi_1$ , we change Eq. (4) to a  $3 \times 3$  matrix spectral problem

$$\Psi_x = U(u, \lambda)\Psi, \quad (18)$$

$$U(u, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda u & 0 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \quad (19)$$

Apparently, the Gateaux derivative matrix  $U_*(\xi)$  of the spectral matrix  $U$  in the direction  $\xi \in C^\infty(\mathbb{R})$  at point  $u$  is

$$U_*(\xi) \triangleq \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} U(u + \epsilon\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda\xi & 0 & 0 \end{pmatrix} \quad (20)$$

which is obviously an injective homomorphism, i.e.  $U_*(\xi) = 0 \Leftrightarrow \xi = 0$ .

For any given  $C^\infty$ -function  $G$ , we construct the following  $3 \times 3$  matrix equation with respect to  $V = V(G)$

$$V_x - [U, V] = U_*(KG - \lambda^2 JG). \quad (21)$$

**Theorem 1** For the spectral problem (18) and an arbitrary  $C^\infty$ -function  $G$ , the matrix equation (21) has the following solution

$$V = \lambda \begin{pmatrix} -G'' - 3\lambda\partial^{-2}\Upsilon G & 3(G' + \lambda\partial^{-3}\Upsilon G) & -6G \\ -G''' - 3\lambda\partial^{-1}uG' & 2G'' & 3(-G' + \lambda\partial^{-3}\Upsilon G) \\ -G'''' - 3\lambda^2u\partial^{-3}\Upsilon G & G''' - 3\lambda\partial^{-1}uG' & -G'' + 3\lambda\partial^{-2}\Upsilon G \end{pmatrix}, \quad (22)$$

where  $\partial = \partial_x = \frac{\partial}{\partial x}$ ,  $\Upsilon = u\partial + 2\partial u$ , and the superscript ' means the derivative in  $x$ . Therefore,  $J = -3\Upsilon^*\partial^{-3}\Upsilon$  ( $\Upsilon^*$  is the conjugate of  $\Upsilon$ ).

**Proof:** Set

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

and substitute this into Eq. (21). This is a overdetermined equation. Using some calculation techniques [19], we obtain the following results:

$$\begin{aligned} V_{11} &= -\lambda G'' - 3\lambda^2\partial^{-2}\Upsilon G, \\ V_{12} &= 3(\lambda G' + \lambda^2\partial^{-3}\Upsilon G), \\ V_{13} &= -6\lambda G, \\ V_{21} &= -\lambda G''' - 3\lambda^2\partial^{-1}uG', \\ V_{22} &= 2\lambda G'', \\ V_{23} &= 3\lambda(-G' + \lambda\partial^{-3}\Upsilon G), \\ V_{31} &= -\lambda G'''' - 3\lambda^3u\partial^{-3}\Upsilon G, \\ V_{32} &= \lambda G''' - 3\lambda^2\partial^{-1}uG', \\ V_{33} &= -\lambda G'' + 3\lambda^2\partial^{-2}\Upsilon G, \end{aligned}$$

which completes the proof.

**Theorem 2** Let  $G_0 \in \text{Ker } J$ ,  $G_{-1} \in \text{Ker } K$ , and let each  $G_j$  be given through Eq. (13). Then,

1. each new vector field  $X_k = JG_k$ ,  $k \in \mathbb{Z}$  satisfies the following commutator representation

$$V_{k,x} - [U, V_k] = U_*(X_k), \quad \forall k \in \mathbb{Z}; \quad (23)$$

2. the new hierarchy (14), i.e.

$$u_{t_k} = X_k = JG_k, \quad \forall k \in \mathbb{Z}, \quad (24)$$

possesses the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \quad \forall k \in \mathbb{Z}, \quad (25)$$

where

$$V_k = \sum V(G_j) \lambda^{2(k-j-1)}, \quad \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0, \\ 0, & k = 0, \\ -\sum_{j=k}^{-1}, & k < 0, \end{cases} \quad (26)$$

and  $V(G_j)$  is given by Eq. (22) with  $G = G_j$ .

**Proof:**

1. For  $k = 0$ , it is obvious. For  $k < 0$ , we have

$$\begin{aligned} V_{k,x} - [U, V_k] &= -\sum_{j=k}^{-1} \left( V_x(G_j) - [U, V(G_j)] \right) \lambda^{2(k-j-1)} \\ &= -\sum_{j=k}^{-1} U_* (KG_j - \lambda^2 KG_{j-1}) \lambda^{2(k-j-1)} \\ &= U_* \left( \sum_{j=k}^{-1} KG_{j-1} \lambda^{2(k-j)} - KG_j \lambda^{2(k-j-1)} \right) \\ &= U_* (KG_{k-1} - KG_{-1} \lambda^{2k}) \\ &= U_* (KG_{k-1}) \\ &= U_*(X_k). \end{aligned}$$

For the case of  $k > 0$ , it is similar to prove.

2. Noticing  $U_{t_k} = U_*(u_{t_k})$ , we obtain

$$U_{t_k} - V_{k,x} + [U, V_k] = U_*(u_{t_k} - X_k).$$

The injectiveness of  $U_*$  implies item 2 holds.

From **Theorem 2**, we immediately obtain the following corollary.

**Corollary 1** *The new hierarchy (14) has Lax pair:*

$$\psi_{xxx} = -\lambda u \psi, \quad (27)$$

$$\psi_{t_k} = \sum \lambda^{2(k-j)-1} \left[ -6G_j \psi_{xx} + 3(G'_j + \lambda \partial^{-3} \Upsilon G_j) \psi_x - (G''_j + 3\lambda \partial^{-2} \Upsilon G_j) \psi \right], \quad (28)$$

where the related symbols are the same as Theorem 2 and Theorem 1.

So, all equations in the hierarchy (14) have the Lax pair and are therefore integrable. In particular, we have the following special cases.

- When we choose  $G_{-1} = \frac{1}{6}$ , equation (15) has the following Lax pair:

$$\Psi_x = U(u, \lambda)\Psi, \quad (29)$$

$$\Psi_t = V(u, \lambda)\Psi, \quad (30)$$

where  $u = v_{xx}$ ,  $U(u, \lambda)$  is defined by Eq. (19), and  $V(u, \lambda)$  is given by

$$V(u, \lambda) = \begin{pmatrix} v_x & -v & \lambda^{-1} \\ 0 & 0 & -v \\ \lambda v u & 0 & -v_x \end{pmatrix}. \quad (31)$$

Apparently, Lax pair (29) and (30) is equivalent to

$$\psi_{xxx} = -\lambda u \psi, \quad (32)$$

$$\psi_t = \lambda^{-1} \psi_{xx} - v \psi_x + v_x \psi. \quad (33)$$

which is a limit form in Ref. [6] when  $\alpha$  goes to  $\infty$ .

- In a similar way, choosing  $G_0 = u^{-\frac{2}{3}}$  gives the Lax pair of equation (16), i.e.  $u_t = (u^{-\frac{2}{3}})_{xxxxx}$

$$\psi_{xxx} = -\lambda u \psi, \quad (34)$$

$$\psi_t = -6\lambda u^{-\frac{2}{3}} \psi_{xx} + 3\lambda (u^{-\frac{2}{3}})_x \psi_x - \lambda (u^{-\frac{2}{3}})_{xx} \psi. \quad (35)$$

This Lax pair is different from/inequivalent to the result in Ref. [13].

- Furthermore, through choosing  $G_0 = \frac{(u^{-\frac{1}{3}})_{xx} - 2(u^{-\frac{1}{6}})_x^2}{u}$ , we find that the new equation (2) has the Lax pair:

$$\psi_{xxx} = -\lambda u \psi, \quad (36)$$

$$\psi_t = -6\lambda G_0 \psi_{xx} + 3\lambda (G'_0 + 3\lambda u^{-\frac{1}{3}}) \psi_x - \lambda (G''_0 + 9\lambda (u^{-\frac{1}{3}})_{xx}) \psi. \quad (37)$$

## 4 Nonlinearized $6N$ -dimensional integrable system from spectral problems

To discuss the solution of the hierarchy (14), we use the constrained method which leads finite dimensional integrable system to the PDEs (14). Because Eq. (4)/(18) is a 3rd order eigenvalue problem, we have to investigate itself together with its adjoint problem when we adopt the nonlinearized procedure [5]. Ma and Strampp [17] ever studied the AKNS and its adjoint problem, a  $2 \times 2$  case, by using the so-called symmetry constraint method. Now, we are discussing a  $3 \times 3$  problem related to the hierarchy (14).

Let us return to the spectral problem (18) and consider its adjoint problem (5), and change it to the following matrix form

$$\Psi_x^* = \begin{pmatrix} 0 & 0 & u\lambda \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \Psi^*, \quad \Psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}, \quad (38)$$

where  $\psi^* = \psi_3^*$ .

Let  $\lambda_j$  ( $j = 1, \dots, N$ ) be  $N$  distinct spectral values of (18) and (38), and  $q_{1j}, q_{2j}, q_{3j}$  and  $p_{1j}, p_{2j}, p_{3j}$  be the corresponding spectral functions, respectively. Then we have

$$\begin{aligned} q_{1x} &= q_2, \\ q_{2x} &= q_3, \\ q_{3x} &= -u\Lambda q_1; \end{aligned} \quad (39)$$

and

$$\begin{aligned} p_{1x} &= u\Lambda p_3, \\ p_{2x} &= -p_1, \\ p_{3x} &= -p_2, \end{aligned} \quad (40)$$

where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ ,  $q_k = (q_{k1}, q_{k2}, \dots, q_{kN})^T$ ,  $p_k = (p_{k1}, p_{k2}, \dots, p_{kN})^T$ ,  $k = 1, 2, 3$ .

Let us consider the above systems in the whole symplectic space  $(\mathbb{R}^{6N}, dp \wedge dq)$ . We directly impose the following constraint:

$$u^{-\frac{2}{3}} = \sum_{j=1}^N \nabla \lambda_j, \quad (41)$$

where  $\nabla \lambda_j = \lambda_j q_{1j} p_{3j}$  is the functional gradient of  $\lambda_j$  for spectral problems (18) and (38). Then Eq. (41) is saying

$$u = \langle \Lambda q_1, p_3 \rangle^{-\frac{3}{2}} \quad (42)$$

which composes a constraint in the whole space  $\mathbb{R}^{6N}$ . Under this constraint, Eq. (39) and its adjoint (40) are cast in a Hamiltonian canonical form in  $\mathbb{R}^{6N}$ :

$$\begin{aligned} q_x &= \{q, H^+\}, \\ p_x &= \{p, H^+\}, \end{aligned} \quad (43)$$

with the Hamiltonian

$$H^+ = \langle q_2, p_1 \rangle + \langle q_3, p_2 \rangle + \frac{2}{\sqrt{\langle \Lambda q_1, p_3 \rangle}}, \quad (44)$$

where  $p = (p_1, p_2, p_3)^T$ ,  $q = (q_1, q_2, q_3)^T \in \mathbb{R}^{6N}$ ,  $\langle \cdot, \cdot \rangle$  stands for the standard inner product in  $\mathbb{R}^N$ , and we modify the usual Poisson bracket of two functions  $F_1, F_2$  as follows:

$$\{F_1, F_2\} = \sum_{i=1}^3 \left( \left\langle \frac{\partial F_1}{\partial q_i}, \frac{\partial F_2}{\partial p_i} \right\rangle - \left\langle \frac{\partial F_1}{\partial p_i}, \frac{\partial F_2}{\partial q_i} \right\rangle \right) \quad (45)$$

which is still antisymmetric, bilinear and satisfies the Jacobi identity.

To see the integrability of the system (43), we take into account of the time part  $\Psi_t = V_k \Psi$  and its adjoint  $\Psi_t^* = -V_k^T \Psi^*$ , where  $V_k$  is defined by  $V_k = \sum_{j=0}^{k-1} V(G_j) \lambda^{2(k-j-1)}$ , and  $V(G_j)$  is given by Eq. (22) with  $G = G_j$ .

Let us first look at  $V_1$  case. Then the corresponding time part is:

$$\Psi_t = \lambda \begin{pmatrix} -(u^{-\frac{2}{3}})_{xx} & 3(u^{-\frac{2}{3}})_x & -6u^{-\frac{2}{3}} \\ -(u^{-\frac{2}{3}})_{xxx} + 6\lambda u^{\frac{1}{3}} & 2(u^{-\frac{2}{3}})_{xx} & -3(u^{-\frac{2}{3}})_x \\ -(u^{-\frac{2}{3}})_{xxxx} & (u^{-\frac{2}{3}})_{xxx} + 6\lambda u^{\frac{1}{3}} & -(u^{-\frac{2}{3}})_{xx} \end{pmatrix} \Psi, \quad (46)$$

and its adjoint part is:

$$\Psi_t^* = \lambda \begin{pmatrix} (u^{-\frac{2}{3}})_{xx} & (u^{-\frac{2}{3}})_{xxx} + 6\lambda u^{\frac{1}{3}} & -(u^{-\frac{2}{3}})_{xxxx} \\ -3(u^{-\frac{2}{3}})_x & -2(u^{-\frac{2}{3}})_{xx} & -(u^{-\frac{2}{3}})_{xxx} - 6\lambda u^{\frac{1}{3}} \\ 6u^{-\frac{2}{3}} & 3(u^{-\frac{2}{3}})_x & (u^{-\frac{2}{3}})_{xx} \end{pmatrix} \Psi^*. \quad (47)$$

Noticing the following relations

$$\begin{aligned} u^{\frac{1}{3}} &= \langle \Lambda q_1, p_3 \rangle^{-\frac{1}{2}}, \\ (u^{-\frac{2}{3}})_x &= \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_1, p_2 \rangle, \\ (u^{-\frac{2}{3}})_{xx} &= \langle \Lambda q_3, p_3 \rangle + \langle \Lambda q_1, p_1 \rangle - 2 \langle \Lambda q_2, p_2 \rangle, \\ (u^{-\frac{2}{3}})_{xxx} &= 3 \left( \langle \Lambda q_2, p_1 \rangle - \langle \Lambda q_3, p_2 \rangle \right), \\ (u^{-\frac{2}{3}})_{xxxx} &= 6 \langle \Lambda q_3, p_1 \rangle + 3 \langle \Lambda q_1, p_3 \rangle^{-\frac{3}{2}} \left( \langle \Lambda^2 q_1, p_2 \rangle + \langle \Lambda^2 q_2, p_3 \rangle \right), \end{aligned}$$

we obtain the nonlinearizations of the time parts (46) and (47), and cast the nonlinearized systems into canonical Hamiltonian system in  $\mathbb{R}^{6N}$ :

$$\begin{aligned} q_{t_1} &= \{q, F_1^+\}, \\ p_{t_1} &= \{p, F_1^+\}, \end{aligned} \quad (48)$$

with the Hamiltonian

$$\begin{aligned} F_1^+ &= -\frac{1}{2} \left( \langle \Lambda q_1, p_1 \rangle + \langle \Lambda q_3, p_3 \rangle \right)^2 + 2 \langle \Lambda q_2, p_2 \rangle \left( \langle \Lambda q_1, p_1 \rangle + \langle \Lambda q_3, p_3 \rangle - \langle \Lambda q_2, p_2 \rangle \right) \\ &\quad + 3 \left( \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_1, p_2 \rangle \right) \left( \langle \Lambda q_2, p_1 \rangle - \langle \Lambda q_3, p_2 \rangle \right) - 6 \langle \Lambda q_1, p_3 \rangle \langle \Lambda q_3, p_1 \rangle \\ &\quad + \frac{6}{\sqrt{\langle \Lambda q_1, p_3 \rangle}} \left( \langle \Lambda^2 q_1, p_2 \rangle + \langle \Lambda^2 q_2, p_3 \rangle \right). \end{aligned} \quad (49)$$

A direct computation leads to the following theorem.

### Theorem 3

$$\{H^+, F_1^+\} = 0, \quad (50)$$

that is, two Hamiltonian flows commute in  $\mathbb{R}^{6N}$ .

Furthermore, for general case  $V_k$ ,  $k > 0$ ,  $k \in \mathbb{Z}$ , we consider the following Hamiltonian functions

$$\begin{aligned} F_k^+ = & -\frac{1}{2} \sum_{j=0}^{k-1} \left( \langle \Lambda^{2j+1} q_1, p_1 \rangle + \langle \Lambda^{2j+1} q_3, p_3 \rangle \right) \left( \langle \Lambda^{2(k-j)-1} q_1, p_1 \rangle + \langle \Lambda^{2(k-j)-1} q_3, p_3 \rangle \right) \\ & + 2 \sum_{j=0}^{k-1} \langle \Lambda^{2j+1} q_2, p_2 \rangle \left( \langle \Lambda^{2(k-j)-1} q_1, p_1 \rangle + \langle \Lambda^{2(k-j)-1} q_3, p_3 \rangle - \langle \Lambda^{2(k-j)-1} q_2, p_2 \rangle \right) \\ & + 3 \sum_{j=0}^{k-1} \left( \langle \Lambda^{2j+1} q_2, p_3 \rangle - \langle \Lambda^{2j+1} q_1, p_2 \rangle \right) \left( \langle \Lambda^{2(k-j)-1} q_2, p_1 \rangle - \langle \Lambda^{2(k-j)-1} q_3, p_2 \rangle \right) \\ & - 6 \sum_{j=0}^{k-1} \langle \Lambda^{2j+1} q_1, p_3 \rangle \langle \Lambda^{2(k-j)-1} q_3, p_1 \rangle \\ & - \frac{3}{2} \sum_{j=0}^k \left( \langle \Lambda^{2j} q_1, p_1 \rangle - \langle \Lambda^{2j} q_3, p_3 \rangle \right) \left( \langle \Lambda^{2(k-j)} q_1, p_1 \rangle - \langle \Lambda^{2(k-j)} q_3, p_3 \rangle \right) \\ & - 3 \sum_{j=0}^k \left( \langle \Lambda^{2j} q_2, p_3 \rangle + \langle \Lambda^{2j} q_1, p_2 \rangle \right) \left( \langle \Lambda^{2(k-j)} q_2, p_1 \rangle + \langle \Lambda^{2(k-j)} q_3, p_2 \rangle \right) \\ & + 3H^+ \left( \langle \Lambda^{2k} q_1, p_2 \rangle + \langle \Lambda^{2k} q_2, p_3 \rangle \right). \end{aligned} \quad (51)$$

Then through a lengthy calculation, we find

$$\{H^+, F_k^+\} = 0, \{F_l^+, F_k^+\} = 0, \quad k, l = 1, 2, \dots \quad (52)$$

That is,

**Theorem 4** *All canonical Hamiltonian flows  $(F_k^+)$  commute with the Hamiltonian system (43). In particular, the Hamiltonian systems (43) and (48) are compatible and therefore integrable in the Liouville sense.*

**Remark 1** *In the proof of this Theorem, we use the following two facts:  $\langle q_1, p_2 \rangle + \langle q_2, p_3 \rangle = c_1$ , and  $\langle q_1, p_1 \rangle - \langle q_3, p_3 \rangle = c_2$ . They always hold along  $x$ -flow in the whole  $\mathbb{R}^{6N}$ . Here  $c_1, c_2$  are two constants.*

**Remark 2** *In fact, the involutive functions  $F_k^+$  are generated from the nonlinearization of the time part  $\Psi_t = V_k \Psi$  and its adjoint part  $\Psi_t^* = -V_k^T \Psi^*$  under the constraint (41), where  $V_k$  is defined by  $V_k = \sum_{j=0}^{k-1} V(G_j) \lambda^{2(k-j-1)}$ , and  $V(G_j)$  is given*

by Eq. (22) with  $G = G_j$ . In this calculation process, we use the following equalities:

$$\begin{aligned}
G_j &= -\langle \Lambda^{2j+1} q_1, p_3 \rangle, \quad j = 0, 1, 2, \dots \\
G'_j &= \langle \Lambda^{2j+1} q_2, p_3 \rangle - \langle \Lambda^{2j+1} q_1, p_2 \rangle, \\
G''_j &= \langle \Lambda^{2j+1} q_3, p_3 \rangle + \langle \Lambda^{2j+1} q_1, p_1 \rangle - 2 \langle \Lambda^{2j+1} q_2, p_2 \rangle, \\
G'''_j &= 3 \left( \langle \Lambda^{2j+1} q_2, p_1 \rangle - \langle \Lambda^{2j+1} q_3, p_2 \rangle \right), \\
G''''_j &= 6 \langle \Lambda^{2j+1} q_3, p_1 \rangle + 3 \langle \Lambda q_1, p_3 \rangle^{-\frac{3}{2}} \left( \langle \Lambda^{2j+2} q_1, p_2 \rangle + \langle \Lambda^{2j+2} q_2, p_3 \rangle \right), \\
\partial^{-1} m G'_j &= \langle \Lambda^{2j} q_3, p_2 \rangle + \langle \Lambda^{2j} q_2, p_1 \rangle, \\
\partial^{-2} \Upsilon G_j &= \langle \Lambda^{2j} q_1, p_1 \rangle - \langle \Lambda^{2j} q_3, p_3 \rangle, \\
\partial^{-3} \Upsilon G_j &= - \left( \langle \Lambda^{2j} q_1, p_2 \rangle + \langle \Lambda^{2j} q_2, p_3 \rangle \right).
\end{aligned}$$

## 5 Parametric solution

Since the Hamiltonian flows  $(H^+)$  and  $(F_k^+)$  are completely integrable in  $\mathbb{R}^{6N}$  and their Poisson brackets  $\{H^+, F_k^+\} = 0$  ( $k = 1, 2, \dots$ ), their phase flows  $g_{H^+}^x$ ,  $g_{F_k^+}^{t_k}$  commute [3]. Thus, we can define their compatible solution as follows:

$$\begin{pmatrix} q(x, t_k) \\ p(x, t_k) \end{pmatrix} = g_{H^+}^x g_{F_k^+}^{t_k} \begin{pmatrix} q(x^0, t_k^0) \\ p(x^0, t_k^0) \end{pmatrix}, \quad k = 1, 2, \dots, \quad (53)$$

where  $x^0$ ,  $t_k^0$  are the initial values of phase flows  $g_{H^+}^x$ ,  $g_{F_k^+}^{t_k}$ .

**Theorem 5** Let  $q(x, t_k) = (q_1, q_2, q_3)^T$ ,  $p(x, t_k) = (p_1, p_2, p_3)^T$  be the common solution of the two commutable Hamiltonian flows  $(H_+)$  and  $(F_k^+)$  in  $\mathbb{R}^{6N}$ . Then

$$u = \frac{1}{\sqrt{\langle \Lambda q_1(x, t_k), p_3(x, t_k) \rangle^3}}, \quad (54)$$

satisfies the positive order hierarchy

$$u_{t_k} = J \mathcal{L}^k \cdot u^{-\frac{2}{3}}, \quad k = 1, 2, \dots, \quad (55)$$

where the operators  $\mathcal{L} = J^{-1}K$ ,  $J$ ,  $K$  are given by Eqs. (10) and (9), respectively.

**Proof:** Direct computation completes this proof.

**Theorem 6** Let  $p(x, t), q(x, t)$  ( $p(x, t) = (p_1, p_2, p_3)^T$ ,  $q(x, t) = (q_1, q_2, q_3)^T$ ) be the common solution of the two integrable commutable flows (43) and (48), then

$$u = \frac{1}{\sqrt{\langle \Lambda q_1(x, t), p_3(x, t) \rangle^3}}, \quad (56)$$

satisfies the equation:

$$u_t = \partial_x^5 u^{-\frac{2}{3}}. \quad (57)$$

**Proof:** Doing five times derivatives in  $x$  on both sides of Eq. (56), we obtain

$$\partial_x^5 u^{-\frac{2}{3}} = 9u \left( \langle \Lambda^2 q_3, p_3 \rangle - \langle \Lambda^2 q_1, p_1 \rangle \right) + 3u_x \left( \langle \Lambda^2 q_1, p_2 \rangle + \langle \Lambda^2 q_2, p_3 \rangle \right), \quad (58)$$

where

$$u_x = -\frac{3}{2}u \frac{\left( \langle \Lambda^2 q_1, p_2 \rangle + \langle \Lambda^2 q_2, p_3 \rangle \right) \left( \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_1, p_2 \rangle \right)}{\langle \Lambda q_1, p_3 \rangle}.$$

On the other hand, doing the derivative in  $t$  on the both sides of Eq. (56) yields

$$\begin{aligned} u_t &= -\frac{3}{2}u \frac{\langle \Lambda p_3, \dot{q}_1 \rangle + \langle \Lambda q_1, \dot{p}_3 \rangle}{\langle \Lambda q_1, p_3 \rangle} \\ &= -\frac{3}{2}u \frac{\left\langle \Lambda p_3, \frac{\partial F_1^+}{\partial p_1} \right\rangle - \left\langle \Lambda q_1, \frac{\partial F_1^+}{\partial q_3} \right\rangle}{\langle \Lambda q_1, p_3 \rangle}. \end{aligned}$$

Substituting the expression of  $F_1^+$  into the above equality and calculating, we find that this final result is the same as the right hand side of Eq. (58), which completes the proof.

## 6 Traveling wave solutions

**First,** Let us compute traveling wave solution for equation (3). Set  $u = f(\xi)$ ,  $\xi = x - ct$  ( $c$  is some constant speed), then after substituting this setting into equation (3) we obtain

$$-cf''' + 3f''f' + f'''f = 0,$$

i.e.

$$(f^2 - 2cf)''' = 0.$$

Therefore,

$$(f - c)^2 = A\xi^2 + B\xi + C, \quad \forall A, B, C \in \mathbb{R}. \quad (59)$$

So, the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$  has the following traveling wave solution

$$u(x, t) = c \pm \sqrt{A(x - ct)^2 + B(x - ct) + C}. \quad (60)$$

Let us discuss several special cases:

- When  $c = 0$ , we get stationary solution

$$u(x) = \pm \sqrt{Ax^2 + Bx + C}, \quad \forall A, B, C \in \mathbb{R}, \quad (61)$$

which can be a straight line, circle, ellipse, parabola, and hyperbola according to different choices of constants  $A, B, C$ .

- When  $c \neq 0$  and  $A \neq 0$ , then we have

$$u(x, t) = c \pm \sqrt{A\left(x - ct + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A}}, \quad \forall A, B, C \in \mathbb{R}, \quad (62)$$

therefore if  $4AC - B^2 = 0$  this solution becomes

$$u(x, t) = c \pm \sqrt{A}\left|x - ct + \frac{B}{2A}\right|, \quad \forall A > 0, B \in \mathbb{R} \quad (63)$$

For example, setting  $c = 1$ ,  $A = 1$ ,  $B = 0$  yields

$$u(x, t) = 1 - |x - t|, \quad (64)$$

and

$$u(x, t) = 1 + |x - t|. \quad (65)$$

The former looks like a compacton solution [20, 8]. The latter is a “V”-type solution.

- When  $c \neq 0$  and  $A = 0$ , then we have

$$u(x, t) = c \pm \sqrt{B(x - ct) + C}, \quad \forall B, C \in \mathbb{R}, \quad (66)$$

which is a parabolic traveling wave solution if  $B \neq 0$  and becomes a constant solution if  $B = 0$ . For example, the following

$$u(x, t) = 1 + \sqrt{x - t}, \quad x - t \geq 0, \quad (67)$$

and

$$u(x, t) = 1 - \sqrt{x - t}, \quad x - t \geq 0. \quad (68)$$

are two special solutions.

So, the 3rd-order equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$  has the continuous traveling wave solution (60).

The Gaussian initial solution of this 3rd-order PDE is stable from  $t = 0$  to  $t = 64$  (see Figure 1). But after  $t = 64$  this solution is not stable (see Figures 2 - 5).

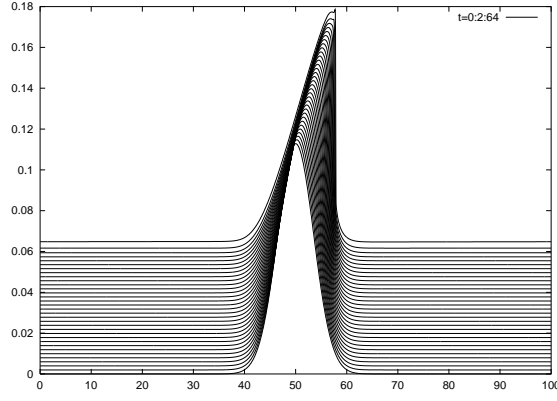


Figure 1: Stable period from  $t = 0$  to  $t = 64$  for the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$  under the Gaussian initial condition. This figure is very like the Burgers case  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u - \epsilon u_{xxxx} = 0$  which is formed through adding small viscosity term  $\epsilon u_{xxxx}$  to the equation. For instance, when  $\epsilon = 0.01$ , the equation has Figure 6.

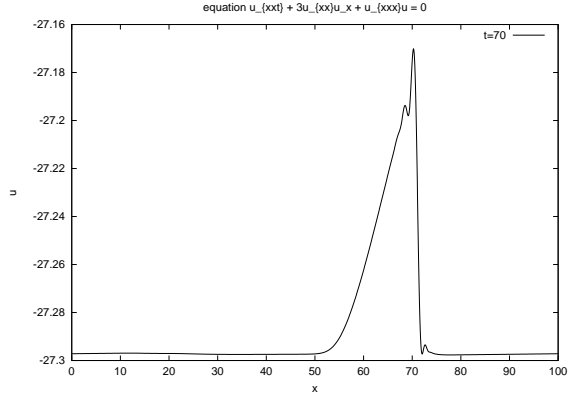


Figure 2: Solution shape at  $t = 70$  for the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ .

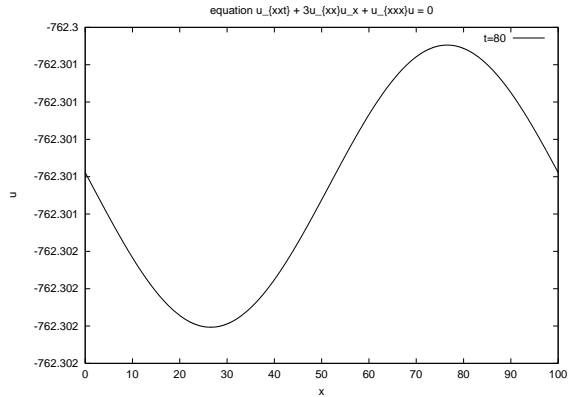


Figure 3: Solution shape at  $t = 80$  for the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ .

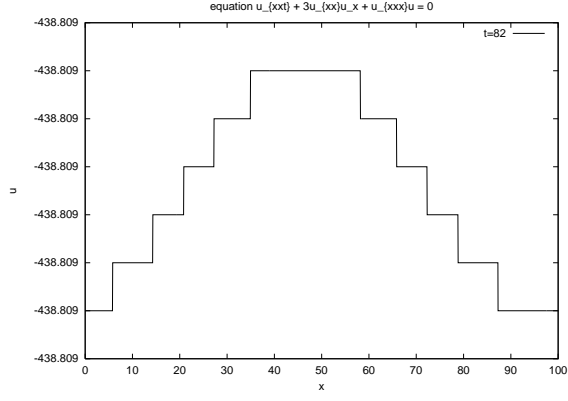


Figure 4: Solution shape at  $t = 82$  for the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ .

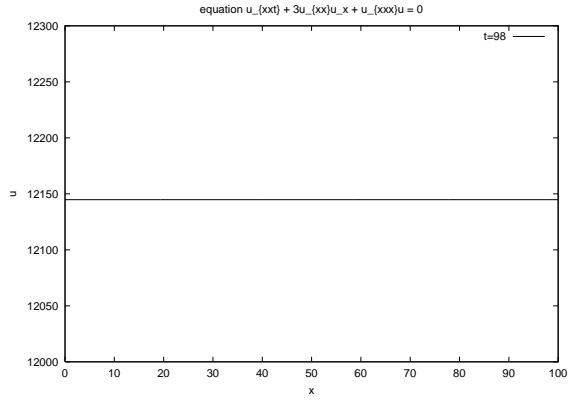


Figure 5: Solution shape at  $t = 98$  for the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ .

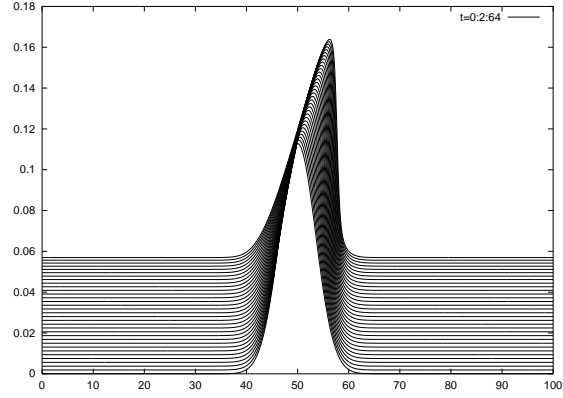


Figure 6: Stable solution for the equation  $u_{xxt} + 3u_{xx}u_x + u_{xxx}u - \epsilon u_{xxxx} = 0$ ,  $\epsilon = 0.01$  under the Gaussian initial condition. This figure is almost same as Figure 1. But, for large  $\epsilon$  they are quite different, and for negative  $\epsilon$ , the corresponding solution blows up.

**Second**, we give traveling wave solution for the 5th-order equation (1). Set  $u = \xi^{-\gamma}$ ,  $\xi = x - ct$  ( $c$  is some constant speed to be determined), then after substituting this setting into equation (1) we obtain

$$\gamma = \frac{12}{5}, \quad c = -\frac{336}{625}. \quad (69)$$

So, the 5th-order equation (1) has the following traveling wave solution

$$u = \left(x + \frac{336}{625}t\right)^{-\frac{12}{5}}. \quad (70)$$

Although at each time the solution (70) has singular point at  $x = -\frac{336}{625}t$ , this 5th-order PDE has the smooth and stable traveling wave solution under the Gaussian initial condition (see Figure 7).

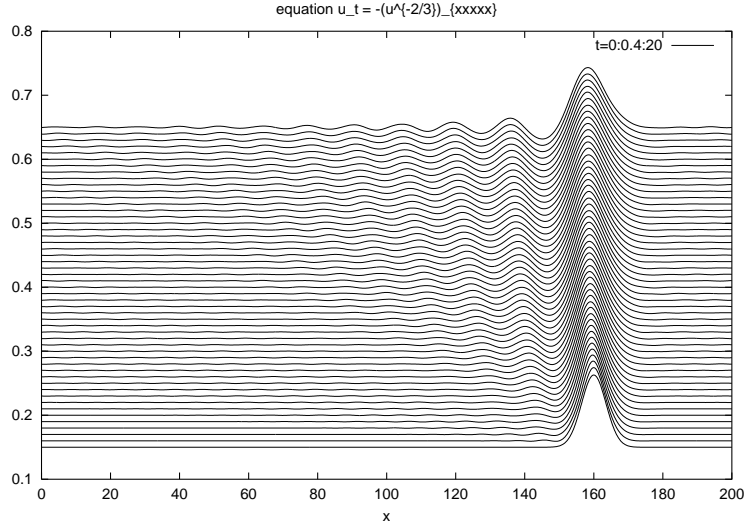


Figure 7: This is the stable solution for the 5th-order equation  $u_t = \partial_x^5 u^{-2/3}$  under the Gaussian initial condition.

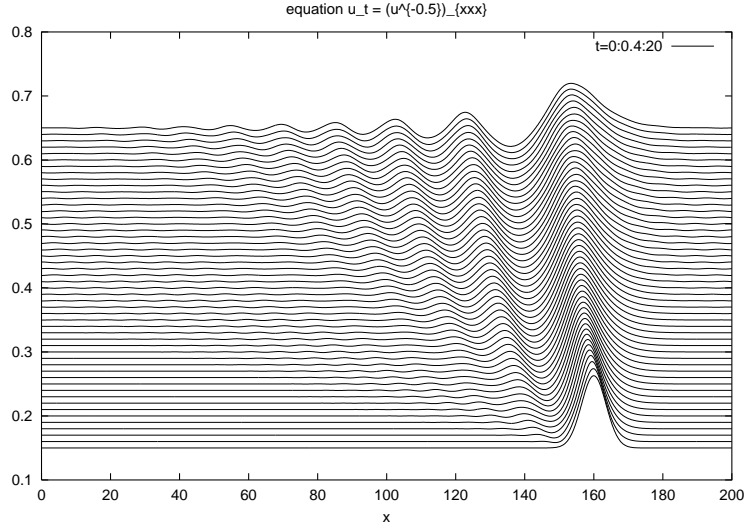


Figure 8: This is the stable solution for the HD equation  $u_t = \partial_x^3 u^{-1/2}$  under the Gaussian initial condition.

So, the figure 7 of the equation  $u_t = \partial_x^5 u^{-2/3}$  has a slight difference from the figure 8 of the HD equation  $u_t = \partial_x^3 u^{-1/2}$ .

**Third**, we give traveling wave solution for the new integrable 7th-order equation (2). Set  $u = \xi^{-\gamma}$ ,  $\xi = x - ct$  ( $c$  is some constant speed to be determined), then we have

$$\gamma = \frac{18}{7}, \quad c = \frac{31680}{117649}. \quad (71)$$

So, the 7th-order equation (2) has the following traveling wave solution

$$u = \left(x - \frac{31680}{117649}t\right)^{-\frac{18}{7}}. \quad (72)$$

Furthermore, we propose the following new equation:

$$u_t = \partial_x^l u^{-m/n}, \quad l \geq 1, \quad n \neq 0, \quad m, n \in \mathbb{Z}. \quad (73)$$

This equation has the following traveling wave solution

$$u(x, t) = (x - ct)^{-n(l-1)/(m+n)}, \quad (74)$$

$$c = \frac{m}{n} \prod_{k=1}^{l-1} \left( \frac{m(l-1)}{m+n} - k \right).$$

Apparently, if  $mn + n^2 > 0$  this solution has singularity at  $x_0 = ct_0$  ( $t_0$  is some time), and if  $mn + n^2 < 0$  this solution is a smooth traveling wave solution.

**Remark 3** Here are the special cusp-like traveling wave solutions

$$u(x, t) = (x - \frac{2}{9}t)^{-4/3} \quad (75)$$

and

$$u(x, t) = (x + \frac{336}{625}t)^{-12/5} \quad (76)$$

for the Harry-Dym equation  $u_t = \partial^3(u^{-1/2})$  and the 5th order equation  $u_t = \partial^5(u^{-2/3})$ .

## 7 Conclusions

In section 5, we obtain a parametric solution (56) of the 5th-order equation (1). This parametric solution can not include its traveling wave solution  $u = (x + \frac{336}{625}t)^{-\frac{12}{5}}$  because the constrained relation

$$\langle \Lambda q_1(x, t), p_3(x, t) \rangle = \left( x + \frac{336}{625}t \right)^{\frac{8}{5}}$$

does not hold, where

$$\begin{aligned} \partial_x^3 q_1 &= - \left( x + \frac{336}{625}t \right)^{-\frac{12}{5}} \Lambda q_1, \\ \partial_x^3 p_3 &= \left( x + \frac{336}{625}t \right)^{-\frac{12}{5}} \Lambda p_3. \end{aligned}$$

Traveling wave solutions  $u = (x + \frac{336}{625}t)^{-\frac{12}{5}}$  for equation  $u_t = \partial^5 u^{-2/3}$  and  $u = (x - \frac{31680}{117649}t)^{-\frac{18}{7}}$  for equation  $u_t = \partial_x^5 \frac{(u^{-\frac{1}{3}})_{xx} - 2(u^{-\frac{1}{6}})_x^2}{u}$  are singular at some certain points  $x$  with the different time  $t$ . That is, this singularity travels with the time  $t$ . In the case of  $n(m+n) > 0$ , the traveling wave solution (74) for general equation (73) is also matching this case.

A natural question arises here: is the equation  $u_t = \partial_x^l u^{-m/n}$  integrable for all  $l \geq 1, m, n \in \mathbb{Z}$  or for what kind of  $l \geq 1, m, n \in \mathbb{Z}$  it is integrable? So far, for the cases:  $l = 3, m = 1, n = 2$  and  $l = 5, m = 2, n = 3$ , we know that it is integrable.

The Harry-Dym equation has the cusp-like traveling wave solution  $u(x, t) = (x - \frac{2}{9}t)^{-4/3}$ , but this is not cusp soliton which Wadati described this in Ref. [22], because the traveling wave solution is singular, but the cusp is continuous.

If we consider other constraints between the potential and the eigenfunctions, then we can still get the parametric solutions for other two equations (2) and (3).

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## References

- [1] Ablowitz M J, Kaup D J, Newell A C, Segur H, Nolinear evolution equations of physical significance, *Phys. Rev. Lett.* 31(1973), 125-127.
- [2] Ablowitz M J, Kaup D J, Newell A C, Segur H, *Studies in Appl. Math.* 53(1974), 249-315.
- [3] V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, Berlin, 1978).
- [4] Camassa R, Holm D D, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (1993), 1661-1664.
- [5] Cao C W, Nonlinearization of Lax system for the AKNS hierarchy, *Sci. China A* (in Chinese) 32(1989), 701-707; also see English Edition: Nonlinearization of Lax system for the AKNS hierarchy, *Sci. Sin. A* 33(1990), 528-536.
- [6] Degasperis A, Holm D D, Hone A N W, A new integrable equation with peakon solutions, *NEEDS (2002) Proceedings*, to appear.
- [7] Degasperis A and Procesi M, Asymptotic integrability, in *Symmetry and Perturbation Theory*, edited by A. Degasperis and G. Gaeta, World Scientific (1999) pp.23-37.
- [8] Fringer D, Holm D D, Integrable vs. nonintegrable geodesic soliton behavior, *Physica D* 150(2001), 237-263.
- [9] Gardner C S, Greene J M, Kruskal M D, Miura R M, Method for Solving the Korteweg-de Vries Equation, *Phys. Rev. Lett.* 19(1967), 1095-1097.
- [10] Holm D D, Hone A N W, Note on Peakon Bracket, Private communication, 2002.
- [11] Holm D D, Qiao Z, Integrable hierarchy,  $3 \times 3$  constrained systems, and parametric and peaked stationary solutions, preprint, 2002.
- [12] Kaup D J, On the inverse scattering problem for cubis eigenvalue problems of the class  $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$ , *Stud. Appl. Math.* 62(1980), 189-216.
- [13] Konopelchenko B G, Dubrovsky V G, Some new integrable nonlinear evolution equations in  $2 + 1$  dimensions, *Phys. Lett. A* 102(1984), 15-17.
- [14] Kuperschmidt B A, A super Korteweg-De Vries equation: an integrable system, *Phys. Lett. A* 102(1984), 213-215.

- [15] Korteweg D J, Vries De G, On the change of form long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Phil. Mag.* 39(1895), 422-443.
- [16] Levitan B M, Gasymov M G, Determination of a differential equation by two of its spectra, *Russ. Math. Surveys* 19:2(1964), 1-63.
- [17] Ma W X, Strampp W, An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, *Phys. Lett. A* 185(1994), 277-286.
- [18] Marchenko V A, Certain problems in the theory of second-order differential operators, *Doklady Akad. Nauk SSSR* 72(1950), 457-460 (Russian).
- [19] Qiao Z J, *Finite-dimensional Integrable System and Nonlinear Evolution Equations*, Higher Education Press, PR China, 2002.
- [20] Rosenau P, Hyman J M, Compactons: Solitons with finite wavelength, *Phys. Rev. Lett* 70(1993), 564 – 567.
- [21] Tu G Z, An extension of a theorem on gradients of conserved densities of integrable systems, *Northeast. Math. J.* 6(1990), 26-32.
- [22] Wadati M, Ichikawa Y H, Shimizu T, Cusp soliton of a new integrable nonlinear evolution equation, *Prog. Theor. Phys.* 64(1980), 1959-1967.
- [23] Zakharov V E, Shabat A B, Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, *Sov. Phys. JETP* 34(1972), 62-69.